Chapter - 6 APPLICATION OF DERIVATIVES

STUDY NOTES

- **Rate of change of quantities**

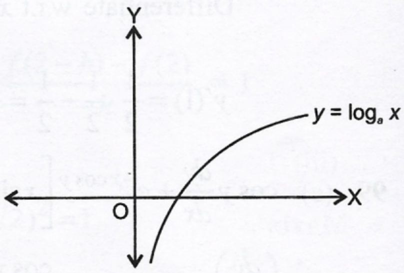
  (i) If $y = f(x)$, $\frac{dy}{dx}$ is the rate of change of y with respect to x.

  (ii) If x is a function of time t, $\frac{dx}{dt}$ measures the rate at which x varies with t.

- **Increasing and decreasing functions** :

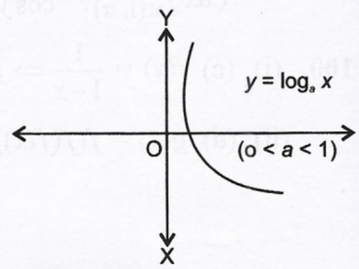
  Increasing function : A function $y = f(x)$ is increasing, if $f(x)$ increases as x increases.

  $f(x_1) > f(x_2)$



  Decreasing Function : A function $y = f(x)$ is a decreasing function, if $f(x)$ decreases as x increases, *i.e.*, $x_1 > x_2$

  $$f(x_1) < f(x_2)$$



  Monotonic Function : Monotonic functions are either Increasing or decreasing function, *i.e.*, $f(x) = \log x$

  $f(x) = \log 2x$ are monotonic functions.

  Condition for monotonic function :

  If 'f' is differentiable real function defined on interval (a, b)

  (i) If $f'(x) > 0\ \forall\ x \in (a, b)$, then $f(x)$ is increasing on (a, b)

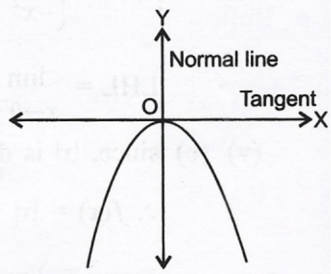
  (ii) If $f'(x) < 0\ f\ \forall\ x \in (a, b)$, then $f(x)$ is decreasing on (a, b)

- **Tangents and Normal :**

  As we know the equation of a straight line passing through a given point $(x_0\, y_0)$ having slope is given as $(y - y_0) = m\ (x - x_0)$

  So, the equation of tangent to this line at the (x_0, y_0) is given as

  $(y - y_0) = \frac{dy}{dx}\ (x - x_0)$ and equation of normal, $(y - y_0) = -\left(\frac{dy}{dx}\right)(x - x_0)$



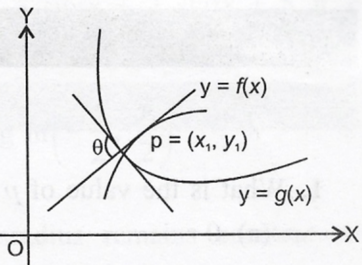
  Different Conditions :

  (i) If tangent is parallel to x-axis, then $\left(\frac{dy}{dx}\right)_{(x_0,y_0)} = 0$

  (ii) If tangent is parallel to y-axis or perpendicular to x-axis, $\left(\frac{dy}{dx}\right)_{(x_0,y_0)} = 0$

  (iii) If the tangent is equally inclined to the axes, then $\frac{dy}{dx} = \tan 45^\circ = \pm 1$

- **Angle of Intersection :**



$$m_1 = \left[\frac{df(x)}{dx}\right]_{(x_1, y_1)} \quad \text{and } m_2 = \left[\frac{dg(x)}{dx}\right]_{(x_1, y_1)}$$

$$\therefore \tan\theta = \left[\frac{m_1 - m_2}{1 + m_1 m_2}\right]$$

- **Orthogonal Curves** :

Two curves are said to be orthogonal curves if they intersect at 90º, and $m_1 m_2 = -1$.

Two curves touch each other if $m_1 = m_2$.

- **Lengths of Tangent, Normal, Subtangent and Subormal :**

(i) Length of tangent $= \left|\dfrac{y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\dfrac{dy}{dx}}\right|$

(ii) Length of normal $= \left|y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\right|$

(iii) Length of subtangent $= \left|\dfrac{y}{\dfrac{dy}{dx}}\right|$

(iv) Length of subnormal $= \left|y\left(\dfrac{dy}{dx}\right)\right|$

- **Maxima and Minima :**

(i) 'f' be a function defined on an Internal I, have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$ for all $x \in$ I.

The number $f(c)$ is the maximum alue of f in I and point c is called point of maximum value.

(ii) f is said to have a minimum value in I, if there exists a point c, in I such that $f(c) < f(x)$ for all $x \in$ I.

The number $f(c)$ in their case is called the minimum value of f in I and the point c in this case is called a point of minimum value of 'f' in I.

- Let 'f' be a real valued function and let c be an interior point in the domain of 'f'. Then,

(i) c is called a point of local maxima if there is an $h > 0$. such that $f(c) \geq f(x)$ for x in $(c - h, c + h)$ $x \neq c$
c is called of local maxima value of f.

(ii) c is called a point of local minima if there is an $h > 0$ such that $f(c) \leq f(x)$ for all x in $(c - h, c + h)$
The value $f(x)$ is called the local minimum value of f.

- **Rolle's Theorem**

If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$

(ii) differentiable in an open interval (a, b), *i.e.,* differentiable at each point in the open interval (a, b)

(iii) $f(a) = f(b)$

$\therefore$ Then, there will be at least one point 'c' in the interval (a, b) such that $f'(c) = 0$.

- **Lagrange's Mean Value Theorem**

If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$

(ii) differentiable in an open interval (a, b). Then there will be at least one point c, where $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

QUESTION BANK

MULTIPLE CHOICE QUESTIONS

1. What is the value of p for which the function $f(x) = p \sin x + \frac{\sin 3x}{3}$ has an extreme at $x = \frac{\pi}{3}$?

(a) 0 (b) 1 (c) – 1 (d) 2

2. The minimum value of $\cos\theta + \cos 2\theta$ is :

(a) –2 (b) 0 (c) $\frac{-9}{8}$ (d) $\frac{-9}{16}$

3. If $x + y = 12$, what is the maximum value of xy ?

(a) 25 (b) 49 (c) 36 (d) 64

4. How many tangents are parallel to the axis for the curve $y = x^2 - 4x + 3$?

(a) 1 (b) 2 (c) 3 (d) none

5. The maximum value of the function : $\log x - x$ is :

(a) –1 (b) 1 (c) 0 (d) ∞

6. The length of subtangent to the curve $x^2y^2 = m^4$ at the point $(-m, m)$ is :

(a) $3m$ (b) $2m$ (c) m (d) $4m$.

7. The rate of change of $\sqrt{x^2+16}$ with respect to x^2 at $x = 3$ is :

(a) $\frac{1}{5}$ (b) $\frac{1}{20}$ (c) $\frac{1}{10}$ (d) $\frac{1}{25}$

8. What is the maximum value of xy with respect to the line $x + y = 8$?

(a) 8 (b) 16 (c) 24 (d) 32

9. What is/are the points on the curve $x^2 + y^2 - 2x - 3 = 0$ where the tangents are parallel to x–axis ?

(a) (1, 2) and (1, –2) (b) (3, 0) and (–3, 0)

(c) (2, 1) and (2, –1) (d) $\left(0,\sqrt{3}\right)$ and $\left(0,-\sqrt{3}\right)$

10. Identify the correct statement.

(a) e^x is a constant function. (b) e^x is increasing function

(c) e^x is decreasing function (d) e^x is neither increasing nor decreasing.

11. If the given function $f(x) = kx^3 - 9x^2 + 9x + 3$ is monotonically increasing in every interval then k is must be :

(a) less than 3 (b) greater than 3 (c) less or equal 3 (d) greater or equal 3.

12. At an extreme point of a function $f(x)$, the tangent to the curve is :

(a) parallel to axis (b) perpendicular to the x-axis

(c) inclined to an angle of 45° to x–axis (d) inclined at an angle 60° to the x-axis.

13. The value of 'a' such that for which $f(x) = \sin x - ax + b$ is decreasing in the interval $(-\infty, \infty)$ is :

(a) $a < 1$ (b) $a > 1$ (c) $a \geq 1$ (d) $a \leq 1$

14. The maximum value of $2x^3 - 3x^2 - 12x + 5$ for $-2 \leq x \leq 2$, when,

(a) $x = -2$ (b) $x = -1$ (c) $x = 2$ (d) $x = 0$

15. The point in the interval $(0, 2\pi)$ where $f(x) = e^x \sin x$ has maximum slope is :

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) 2π

16. A balloon is pumped at the rate of 4 cm^3/s. What is the rate at which its surface area increases when its radius is 4 cm ?

(a) 1 cm^2/s (b) 3 cm^2/s (c) 2 cm^2/s (d) 4 cm^2/s

17. Which of the following is correctly define the behaviour of the curve $f(x) = -\sin^3 x + 3\sin^2 x + 5\ \forall\ x \in \left[-\frac{4}{2}, \frac{\pi}{2}\right]$?

(a) $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$ (b) $f(x)$ is decreasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $f(x)$ is neither increasing nor decreasing (d) $f(x)$ is constant.

18. If the sides and angels of a triangle vary in such a way that its circumradius remains constant. Find $\frac{da}{\cos A} + \frac{db}{\cos b} + \frac{dc}{\cos C}$, where da, db and dc are small increment in the sides, a, b, c respectively.

(a) 1 (b) (c) 0 (d) π

19. The absolute value of maximum $f(x) = \frac{1}{|x-4|+1} + \frac{1}{|x+8|+1}$ is :

(a) 0 (b) 1 (c) $\frac{14}{13}$ (d) $\frac{13}{14}$

20. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall\ x \in (0, \pi/2)$ and $g(x) = f(\sin x) + f(\cos x)$, then $g(x)$ is decreasing in :

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{4}\right)$ (c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

21. Let $f(x)$ be a monotonic polynomial of $2m - 1$ degree, where $m \in$ N, then the equation

$$f(x) + f(3x) + f(5x) + \ldots\ldots f(2m-1) = 2m - 1 \text{ has :}$$

(a) at least one real root (b) exactly one real root

(c) $(2m - 1)$ real root (d) none of these

22. Given that $f'(x) > g'(x)$ for all real x, and $f(0) = g(0)$, then $f(x) < g(x)$ for all n belonging to :

(a) $(0, \infty)$ (b) $(-\infty, 0)$ (c) $(-\infty, \infty)$ (d) none of these

23. The function $f(x) = |ax - b| + C|x|\ \forall\ x \in (-\infty, \infty)$, where $a > 0$, $b > 0$, $c > 0$, assumes it's minimum only at one point if :

(a) $a \neq b$ (b) $a \neq c$ (c) $b \neq c$ (d) $a = b = c$

24. If the equation $x^5 - 10a^3x^2 + b^4 x + c^5 = 0$ has three equal roots, then

(a) $2b^2 - 10a^3 b^2 + c^5 = 0$ (b) $6a^5 + c^5 = 0$ (c) $2c^5 - 10a^3b^2 + b^4c^5 = 0$ (d) $b^4 = 15a$

25. The equation $8x^3 - ax^2 + bx - 1 = 0$ has three real roots in G.P. If, $\lambda_1 \leq a \leq \lambda_2$ then ordered pair (λ_1, λ_2) can be :

(a) $(-2, 2)$ (b) $(24, -2a)$ (c) $(-10, -8)$ (d) none of these

26. If a, b, > 0, let $f(a, b) = \frac{a^3 b}{(a+b)^4}$, then

(a) Absolute maximum value of $f(a, b)$ is $\frac{81}{512}$.

(b) Absolute maximum value of $f(a, b)$ is $\frac{27}{256}$.

(c) The maximum value when $a = b$ is smaller than the minimum value when $a \neq b$.

(d) Maximum value is same for all 'a' and 'b'.

27. If $f(x) = \int_0^x \frac{\sin x}{x}\,dx$, $x > 0$, then

(a) $f(x)$ has local maxima at $x = n\pi \cdot (n = 2k, k \in I^+)$

(b) $f(x)$ has local minima at $x = n\pi \cdot (n = 2k, k \in I^+)$

(c) $f(x)$ has neither maxima nor minima at $x = n\pi . (n \in I^+)$

(d) $f(x)$ has local maxima at $x = n\pi\ (n = 2k + 1, k \in I^+)$

28. The values of parameter 'a' for which the point of minimum of function $f(x) = 1 + a^2x - x^3$ satisfies the inequality $\dfrac{x^2+x+2}{x^2+5x+6} < 0$, are

(a) $\left(2\sqrt{3}, 3\sqrt{3}\right)$ (b) $\left(2\sqrt{3}, -3\sqrt{3}\right)$ (c) $\left(-3\sqrt{3}, -2\sqrt{3}\right)$ (d) $\left(-3\sqrt{3}, 2\sqrt{3}\right)$

29. If $f(x) = (x - \alpha)^n\, g(x)$, then $f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots\ldots f^{n-1}\,\alpha = 0$, where $f(x)$ and $g(x)$ are polynomials. For a polynomial $f(x)$ with rational coefficient, then $f(x)$ touches x-axis at only one point, then the point of touching is :

(a) always a rational number (b) may or may not be a rational number

(c) never a rational number (d) none of these.

30. If $f(x) = x^3 + ax^2 + bx + c$ and $f(-3) = f(2) = 0$ and $f'(-3) < 0$, then the largest value of c is :

(a) -18 (b) -12 (c) -19 (d) -6

31. The function $f(x) = (x^2 + x - 2)(x^2 + 2x - 3)$ has local maxima at

(a) $x = 1$ (b) $x = -1$ (c) $x = 0$ (d) $x = \pi$

32. Let $f : \mathrm{D} \to \mathrm{R}$ and $f(x) = \mathrm{I}_n\mathrm{I}_n\mathrm{I}_n\ldots\mathrm{I}_n\left(x^2 - \dfrac{x}{2} + \dfrac{49}{16} + \cos 4\pi\right)$ then the value (s) of 'n' for which 'f' is onto is/are

(a) only 1 (b) exactly 2 (c) 3 (d) none of these

33. The abscissa of the point on the curve $3y = 6x - 5x^3$, the normal at which passes through origin is :

(a) 1 (b) $\dfrac{1}{3}$ (c) 2 (d) $\dfrac{1}{2}$

34. Let $f : (0, 1) \to \mathrm{R}$ be defined by $f(x) = \dfrac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

(a) f is not invertible on (0, 1) (b) $f \neq f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f(0)}$

(c) $f = f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f'(0)}$ (d) f^{-1} is differentiable on (0, 1)

35. Let f, g and h be real valued function defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = e^{x^2} + e^{-x^2}$ and f, g and h on [0, 1], then

(a) $a = b$ and $c \neq b$ (b) $a \neq c$ and $c \neq b$ (c) $a \neq c$ and $c \neq b$ (d) $a = b = c$

36. Let the function $g : (-\infty, \infty) \to \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \dfrac{\pi}{2}$. Then g is :

(a) even and is strictly increasing in $(0, \infty)$ (b) odd and is strictly decreasing in $(-\infty, \infty)$

(c) odd and is strictly increasing in $(-\infty, \infty)$ (d) neither even nor odd but is strictly increasing $(-\infty, \infty)$

37. The total number of local maxima and local minima of the function :

$$f(x) = \begin{cases} (2+x)^3 & -3 < x \le -1 \\ x^{2/3} & -1 < x < 2 \end{cases}$$

(a) 0 (b) 1 (c) 2 (d) 3

38. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$.

(a) on the left of $x = c$ (b) on the right of $x = c$ (c) at no point (d) at all points

39. If $f(x)$ is a twice, differentiable function and given that $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, then

(a) $f''(x) = 2, \forall\, x \in (1, 3)$ (b) $f''(x) = f'(x) = 5$ for $x \in (2, 3)$

(c) $f''(x) = 3\ \forall\, x \in (2, 3)$ (d) $f''(x) = 2\ \forall\, x \in (1, 3)$

40. The minimum area of triangle formed by the tangent formed by the tangent to the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinates axis is :

(a) ab sq. units (b) $\frac{a^2+b^2}{2}$ sq. units (c) $\frac{(a+b)^2}{2}$ sq. units (d) $\frac{a^2+ab+b^2}{3}$ sq. units.

41. If $y = y(x)$ and it follows the relation $x \cos y + y \cos x = \pi$ then $y''(0)$ is equal to :

(a) 1 (b) -1 (c) π (d) $-\pi$

42. If $f(x)$ is a continuous and differentiable function and $f\left(\frac{1}{n}\right) = 0\ \forall\ n \ge 1$ and $n \in 1$, then

(a) $f(x) = 0, x \in (0, 1]$ (b) $f'(0) = 0 = f''(0), x \in (0, 1)$

(c) $f(0) = 0, f'(0) = 0$ (d) $f(0) = 0$ and $f'(0)$ need not to be zero.

43. If $f(x) = x^{\alpha} \log x$ and $f(0) = 0$, then the value of α, for which Rolle's theorem can be applied in [0, 1] is :

(a) -2 (b) -1 (c) 0 (d) $\frac{1}{2}$

44. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then $(-\infty, \infty)$

(a) $f(x)$ is strictly increasing function (b) $f(x)$ has a local maxima

(c) $f(x)$ is strictly decreasing function (d) $f(x)$ is bounded.

45. If y is a function of x and $\log (x + y) - 2xy = 0$, then the value of $y'(0)$ is equal to :

(a) 1 (b) 2 (c) -1 (d) 0

46. In [0, 1] Lagrange's Mean Value Theorem is not applicable :

(a) $f(x) = \begin{cases} \frac{-1}{2} - x & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & x \ge \frac{1}{2} \end{cases}$ (b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 1 & x = 0 \end{cases}$ (c) $f(x) = x\,|x|$ (d) $f(x) = |x|$

47. Tangent is drawn to ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ where $\left(\theta \in 0, \frac{\pi}{2}\right)$. Then the value of θ such that sum of intercepts on axes made by this tangent to minimum is :

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

48. The length of the longest interval in which the function $3 \sin x - 4 \sin^3 x$ is increasing is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

49. The point (s) on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical is (are) :

(a) $\left(\pm\frac{4}{\sqrt{3},} - 2\right)$ (b) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$ (c) (0, 0) (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

50. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is :

(a) increasing on $\left[\frac{-1}{2}, 1\right]$ (b) decreasing on R (c) increasing on R (d) decreasing on $\left[\frac{-1}{2}, 1\right]$

51. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the co-ordinate axes, lies in the first quadrant. If its area is 2, then the value of b is :

(a) -1 (b) -2 (c) -3 (d) 1

52. If the normal to the curve $y = f(x)$ at (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis then $f'(3)$ is

(a) -1 (b) $\frac{-3}{4}$ (c) $\frac{4}{3}$ (d) 1

53. If the normal to the curve $y = f(x)$ at (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis then $f'(3)$ is :

(a) -1 (b) $\frac{-3}{4}$ (c) $\frac{4}{3}$ (d) 1

54. Let $f(x) = \begin{cases} |x| \text{ for } 0 < |x| \le 2 \\ 1 \text{ for } x = 0 \end{cases}$. Then at $x = 0$, f has

(a) a local maxima (b) a local minima (c) no local maxima (d) no extremum.

55. The function $f(x) = \sin^4 x + \cos^4 x$ increases if :

(a) $0 < x < \frac{\pi}{8}$ (b) $\frac{\pi}{4} < x < \frac{3\pi}{8}$ (c) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (d) $\frac{5\pi}{8} < x < \frac{3\pi}{8}$

56. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x, then the minimum value of f

(a) does not exists because 'f' is bounded (b) is not attain even through f is bounded

(c) is equal to 1 (d) is equal to -1

57. The number of real values of x, where the function $f(x) = \cos x + \cos(\sqrt{2}\,x)$ attains its maximum is :

(a) 0 (b) 1 (c) 2 (d) infinite

58. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \le 1$, then in this interval.

(a) both $f(x)$ and $g(x)$ are increasing (b) both $f(x)$ and $g(x)$ and decreasing.

(c) $f(x)$ is an increasing function (d) $g(x)$ is an increasing function.

59. On the interval [0, 1] the function $x^{25}(1 - x)^{75}$ takes its maximum value at the point :

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

60. The maximum value of the function $f(x) = \sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ in the interval $\left(0, \frac{\pi}{2}\right)$ occurs at :

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

61. $f(x) = \begin{cases} x + 2, & -1 \le x < 0 \\ 1 & x = 0 \\ \frac{x}{2}, & 0 < x \le 1 \end{cases}$ then, on $[-1, 1]$, this function has :

(a) a minimum (b) a maximum

(c) either maxima or minima (d) neither maxima nor minima.

62. Let $f(x)$ be a quadratic function which is positive, for all real x.

If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x.

(a) $g(x) < 0$ (b) $g(x) > 0$ (c) $g(x) = 0$ (d) $g(x) \ge 0$

63. Let f and g be increasing and decreasing functions, respectively $[0, \infty)$ to $[0, \infty)$, Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x) = h(1)$ is :

(a) always zero (b) always negative (c) always positive (d) none of these.

64. The equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2) is :

(a) $x + y = 3$ (b) $x - y = 1$ (c) $x + y = -1$ (d) $x - y = -1$

65. The co-ordinate of the point on the curve $y = \frac{x^2-1}{x^2+1}$, $x > 0$ such that tangent at these point have the greatest slope is :

(a) $\left(\frac{1}{\sqrt{3}}, \frac{-1}{2}\right)$ (b) $\left(\sqrt{3}, \sqrt{2}\right)$ (c) $\left(\sqrt{3}, \frac{1}{2}\right)$ (d) $\left(-\frac{1}{\sqrt{3}}, \frac{-1}{2}\right)$

66. A window of a fixed perimeter is in the form of rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass does. What is the ratio of the sides of the rectangular so that the window transmits the maximum light ?

(a) π (b) 6π (c) $\frac{6}{\pi}$ (d) $\frac{6}{6+\pi}$

67. The equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$ is :

(a) $x + y = 1$ (b) $x - y = -1$ (c) $x - y = 2$ (d) $x + y = -2$

68. Let (h, k) be a fixed point, where $h > 0, k > 0$. A straight line passing through this point cuts the positive direction of the co-ordinate axes at the point P and Q. The minimum area of the triangle OPQ, O being origin is :

(a) $2k$ (b) $2hk$ (c) $2h$ (d) hk

69. A point on the curve $x^2 + 2y^2 = 6$ whose distance from the $x + y = 7$ is minimum is :

(a) (2, 1) (b) (1, 2) (c) (–1, 2) (d) (–2, –1)

70. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$ (c) $a < 0, b < 0$ (d) none of these

71. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$ then which of the following correct ?

(a) $f(x)$ is increasing on [–1, 2] (b) $f(x)$ is continuous on [–1, 3]

(c) $f(x)$ has the maximum value at $x = 2$ (d) all of the above

72. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at $x =$

(a) $\frac{-2}{5}$ (b) $\frac{-2}{3}$ (c) 2 (d) $\frac{2}{3}$

73. Let $h(x) = fx'(f(x)^2 + f(x))^3$ for every real number x. Then

(a) h is increasing whenever f is increasing (b) h is increasing whenever f is decreasing

(c) h is decreasing whenever f is increasing (d) none of these

74. Let $f(x) = \int e^x (x-1)(x-2)\, dx$. Then f decreases in the interval.

(a) $(-\infty, 0)$ (b) $(-1, 2)$ (c) $(-2, -1)$ (d) $(-2, \infty)$

75. The normal to the curve $x^2 + 3xy - xy - 3y^2 = 0$ at (1, 1)

(a) meets the curve again in the second quadrant. (b) meets the curve again in the third quadrant.

(c) meets the curve again in the fourth quadrant. (d) does not meet the curve again.

76. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then

(a) $\alpha = -6, \beta = \frac{1}{2}$ (b) $\alpha = 2, \beta = \frac{-1}{2}$ (c) $\alpha = -6, \beta = \frac{-1}{2}$ (d) $\alpha = 2, \beta = \frac{1}{2}$

77. If f and g are differentiable function in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$ (c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$

78. A value of c for which the conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ *on the* interval [1, 3] is :

(a) $2 \log_3 e$ (b) $\frac{1}{2} \log_e 3$ (c) $\log_3 e$ (d) $\log_e 3$

79. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis is :

(a) $y = 1$ (b) $y = 2$ (c) $y = 3$ (d) $y = 0$

80. Let $f : R \to R$ be defined as $f(x) = \begin{cases} k - 2x \text{ if } & x \le -1 \\ 2x + 3 \text{ if } & x > -1 \end{cases}$

If f has a local minimum at $x = -1$, then a possible value of 'k' is :

(a) 0 (b) $\frac{-1}{2}$ (c) -1 (d) 1

81. The function $f(x) = \tan^{-1} (\sin x + \cos x)$ is an increasing function in :

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(\frac{-\pi}{2}, \frac{\pi}{4}\right)$ (c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

82. The normal to the curve $x = a (1 + \cos \theta)$, $y = a \sin \theta$ at 'θ' is always passes through the fixed point :

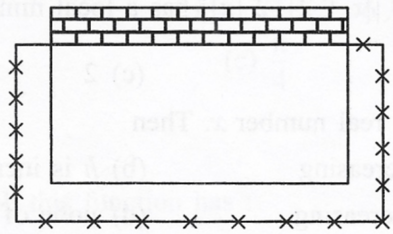
(a) $(a, 0)$ (b) $(0, 0)$ (c) $(0, a)$ (d) (a, a)

83. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ where $a > 0$, attains its maximum and minimum at 'p' and 'q' respectively such that $p^2 = q$ than 'a' is equal to :

(a) 3 (b) 1 (c) 2 (d) $\frac{1}{2}$

INPUT TEXT BASED MCQ's

84. Rammu farmer wants to fence his rectangular garden using bricks and wire. But he has less budget so he decided to build brick wall at the end and wire fence at the remaining three sides, he has 100 feet of wire fencing as shown in the figure,



Answer the following questions :

(i) If Ramu wants to construct a garden using 100 ft of fencing we need to maximise its :

(a) Volume (b) area (c) perimeter (d) length of the side

(ii) If 'b' denote the length of side of garden perpendicular to brick wall and 'l' denotes the length of the side parallel to brick wall, then which of the following relation represents the total amount of fencing wire :

(a) $2l + b = 150$ (b) $2l + b = 100$ (c) $2b + l = 100$ (d) $b + l = 200$

(iii) Area of the garden as a function of b, say A(b) can be represented by :

(a) $100 + 2b^2$ (b) $b - 2b^2$ (c) $100b - 2b^2$ (d) $200 - b^2$

(iv) Maximum value A(x) occurs at x-equals :

(a) 50 ft (b) 30 ft (c) 25 ft (d) 100 ft

(v) The maximum area of the garden is :

(a) 3750 sq. ft (b) 2500 sq. ft (c) 4000 sq. ft (d) 1000 sq ft

ANSWERS

1. (d)	**2.** (c)	**3.** (c)	**4.** (a)	**5.** (a)	**6.** (c)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (b)
11. (b)	**12.** (a)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (b)
21. (b)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (c)
31. (a)	**32.** (c)	**33.** (a)	**34.** (a)	**35.** (d)	**36.** (c)	**37.** (c)	**38.** (a)	**39.** (d)	**40.** (a)
41. (c)	**42.** (c)	**43.** (d)	**44.** (a)	**45.** (a)	**46.** (a)	**47.** (b)	**48.** (a)	**49.** (d)	**50.** (a)
51. (c)	**52.** (d)	**53.** (d)	**54.** (a)	**55.** (b)	**56.** (c)	**57.** (b)	**58.** (c)	**59.** (b)	**60.** (a)
61. (d)	**62.** (b)	**63.** (d)	**64.** (a)	**65.** (a)	**66.** (d)	**67.** (a)	**68.** (b)	**69.** (a)	**70.** (b)
71. (d)	**72.** (b)	**73.** (a)	**74.** (c)	**75.** (c)	**76.** (a)	**77.** (d)	**78.** (a)	**79.** (c)	**80.** (c)
81. (b)	**82.** (a)	**83.** (c)	**84.** (i) (b)	(ii) (c)	(iii) (d)	(iv) (c)	(v) (a)		

Hints to Some Selected Questions

1. (d) $f'(x) = \pi \cos x + \dfrac{3\cos 3x}{3} = \pi \cos x + \cos 3x$

$f(x)$ has an extreme value at $x = \dfrac{\pi}{3} \Rightarrow f'\left(\dfrac{\pi}{3}\right) = 0$

$\Rightarrow p \cos \dfrac{\pi}{3} - \cos \pi = 0 \Rightarrow p\left(\dfrac{1}{2}\right) - 1 = 0 \Rightarrow p = 2$

4. (a) $\dfrac{dy}{dx} = 2x - 4$, at $x = 2$, $\dfrac{dy}{dx} = 0$

$\therefore$ Only one tangent is possible, as slope of tangent parallel to x-axis is zero.

5. (a) $\dfrac{dy}{dx} = \dfrac{1}{x} - 1 \Rightarrow \dfrac{1}{x} - 1 = 0 \Rightarrow x = 1$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-1}{x^2} \Rightarrow$ for $x = 1$, $\dfrac{d^2y}{dx^2} = -$ ve

$\therefore$ The value is maximum for $x = 1 \Rightarrow \log(1) - 1 = -1$.

8. (b) $x + y = 8 \Rightarrow y = 8 - x$

$\Rightarrow c = xy \Rightarrow c = x(8 - x) \Rightarrow \dfrac{dc}{dx} = 8 - 2x \Rightarrow \dfrac{d^2c}{dx^2} = -2$

Put $\dfrac{dc}{dx} = 0$, for maxima and minima $8 - 2x = 0 \Rightarrow x = 4$

$\therefore \left(\dfrac{d^2C}{dx^2}\right)_{x=4} = -2 < 0$ $\therefore$ Max. Value is at $x = 4$, *i.e.*, 16.

9. (a) $2x + 2y\dfrac{dy}{dx} - 2 = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{1-x}{y}$

As per given condition, tangent is parallel to x-axis.

$\therefore \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{1-x}{y} = 0 \Rightarrow x = 1 \Rightarrow (1) + y^2 - 2 - 3 = 0 \Rightarrow y = \pm 2$.

10. (b) $y = f(x) = e^x \Rightarrow \dfrac{dy}{dx} = e^x > 0 \quad x \in \mathrm{R}$

$\therefore f(x) = e^x$ is increasing function.

11. (b) $f(x) = kx^3 - 9x^2 + 9x + 3 \Rightarrow f'(x) = 3kx^2 - 18x + 9 > 0 \Rightarrow \Delta = b^2 - 4ac < 0 \Rightarrow 36 - 12k < 0 \Rightarrow k > 3.$

12. (a) $\frac{dy}{dx} = 0$, For the extreme points. Therefore, tangent to the curve is parallel to x-axis.

13. (b) $f'(x) = \cos x - a$

$\therefore$ function to be decreasing $f'(x) < 0$

$\cos x - a < 0 \Rightarrow \cos x < a$

Value of $\cos x$ lies between -1 and 1

$-1 < a \Rightarrow a > 1$

16. (c) Let 'r' be the radius of balloon.

$\Rightarrow V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi.3r^2\frac{dr}{dt} \Rightarrow 4 = 4\ \pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{16\pi}$

Surface area(s) $= 4\ \pi r^2 \Rightarrow \frac{ds}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt} = 4\pi \times 2 \times 4 \times \frac{1}{16\pi} = 2\text{cm}^2/\text{sec}.$

21. (b) $f(x)$ is monotonic $\Rightarrow f'(x) < 0$ or $f'(x) > 0$

$f'(px) < 0$ or $f'(px) > 0$. $\forall\ x \in R \Rightarrow f(px)$ is also monotonic.

$\therefore f(x) + f(3x) + ... + f(2m - 1)x$ is a monotonic polynomial of odd degree $(2m - 1)$. It can have all real values at only once.

22. (b) Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x) > 0\ \forall\ x \in R\ \therefore\ h(x)$ is increasing function.

$f(0) - g(0) = 0.$

Therefore, $h(x) > 0\ \forall\ x \in (0, \infty)$ and $h(x) < 0\ \forall\ x \in (-\infty, 0)$

26. (b) AM $\geq$ GM, for $(a, a, a, 3b)$ we get, $\Rightarrow \frac{a+a+a+3b}{4} \geq (3a^3 b)^{1/4}$

$\Rightarrow \frac{3}{4}(a + b) \geq (3a^3 b)^{1/4} \Rightarrow \frac{3^4}{256}(a + b)^4 \geq 3a^3 b \Rightarrow \frac{a^3 b}{(a+b)^4} \leq \frac{27}{256}.$

27. (b) $f'(x) = \frac{\sin x}{x}$, for $f'(x) = 0\ \ \frac{\sin x}{x} = 0 \Rightarrow x = n\pi$

$f''(x) = 0;\ \frac{x\cos x - \sin x}{x^2} = f''(n\pi) = \frac{\cos n\pi}{n\pi} < 0$

If $n = 2k - 1$ If $n = 2k\ n \in I^+$

29. (a) If $f(x)$ touches x-axis at only one irrational point, then $f(x) = (x - a)^2 g(x)$, where α is irrational

$\Rightarrow$ Coefficient of $f'(x)$ can't be rational

$\Rightarrow$ for $f(x)$ with rational coefficient, then point of touching is rational.

32. (c) $x^2 - \frac{x}{2} + \frac{49}{16}$ has minimum value 3 at $x = \frac{1}{4}$

$\cos 4\pi x$ has minimum value -1 at $x = \frac{1}{4}$

$\Rightarrow x^2 - \frac{x}{2} + \frac{49}{16} + \cos 4\pi x \geq 2$

35. (d) $f(x) = e^{x^2} + e^{-x^2} \Rightarrow f'(x)\ f(x) = 2x\left(e^{x^2} - e^{-x^2}\right) \geq 0\ x \in [0, 1]$

Clearly for $0 \leq x \leq 1$; $f(x) \geq g(x) \geq h(x)$

$\therefore f(1) = g(1) = h(1) = e + \frac{1}{e}$

and $f(1)$ is the greatest

$a = b = c = e + \frac{1}{e} \Rightarrow a = b = c.$

37. (c) $f(x) = (2 + x)^3, -3 = x^{2/3} \Rightarrow f'(x) = 3(2 + x)^2, -3 = \frac{2}{3x^{\frac{-1}{3}}}, -1$

Clearly $x = -2$ is extremum and $x = -1$, $f(x)$ is not differentiable.

$\therefore$ There are two (2) maximum or minimum points

41. (c) $-x \sin y \frac{dy}{dx} + \cos\ y - y \sin x + \cos x \frac{dy}{dx} = 0$

$\Rightarrow y - (0) = 1.$

Again differentiating and using $y(0) = 1$, $y(0) = \pi$, we get $y\ (0) = \pi$

43. (d) Function must be continuous in the [0, 1]

$\therefore \lim_{x \to 0^+} f(x) = 0 \Rightarrow \lim_{x \to 0^+} \frac{\log x}{x^{-\alpha}}$

Apply L' Hospital Rule

$\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\alpha x^{\alpha - 1}} \Rightarrow \lim_{x \to 0^+} \frac{x^{\alpha}}{-\alpha}$

For the limit to exist $\alpha > 0$ for $\alpha > 0$ $\lim_{x \to 0^+} f(x) = 0$

44. (a) $f'(x) = 3x^2 + 2bx + c,\ b^2 < c$ $\quad$ $D = 4b^2 - 12c = 4(b^2 - 3c) \Rightarrow 4(b^2 - 3c) < 0$

$\Rightarrow b^2 - 3c < 0 \Rightarrow f'(x) > 0.$

45. (a) $\log(x + y) = 2xy$

When $x = 0$, $\log y = 0$, $y = 1$

differentiate both side ; $\frac{dy}{dx} = \left[\frac{2y(x+y)-1}{1-2x(x+y)}\right]$ at (0, 1) $\Rightarrow \frac{dy}{dx} = 1$

46. (a) L.H.D. $= \lim_{h \to 0} \frac{f(a-h) - f(a)}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{2} - \frac{1}{2} + h\right) - \left(\frac{1}{2} - \frac{1}{2}\right)}{h} = 1$

R.H.D. $= \lim_{h \to 0} f\frac{(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{2} - \frac{1}{2} - h\right) - \left(\frac{1}{2} - \frac{1}{2}\right)^2}{h} = 1$

$\therefore$ LHD = RHD. Hence $f(x)$ is not differentiable at $x = \frac{1}{2}$

$\therefore$ It does not follow Lagrange's theorem.

48. (a) Since $3 \sin x - 4 \sin^3 x = \sin 3x$ and $\sin 3x$ is increases when $3x$ can take values from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$ or x takes value from $\frac{-\pi}{6}$ to $\frac{\pi}{6}$.

$\therefore$ The length of longest interval in which $f(x)$ increase is $\frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

52. (d) $\frac{dy}{dx} = f'(x)$ slope of normal $= \frac{-1}{f'(x)}, \frac{-1}{f'(3)}, \tan\frac{3\pi}{4} = -1 \therefore f'(3) = 1.$

53. (d) $\frac{dy}{dx} = f'(x)$, slope of normal = $\frac{-1}{f'(x)}$, $\frac{-1}{f'(3)} = \tan\frac{3\pi}{4} = -1 \Rightarrow f'(3) = 1$.

56. (c) $f'(x) = \frac{(x^2+1)2x-(x-1)(2x)}{(x^2+1)^2} = 0 = \frac{4x}{(x^2+1)^2} = 0, \rightarrow x = 0$

If $x < 0, f'(x) < 0$ and if $x > 0, f'(x) > 0$

$\therefore$ x is point of minima, $\therefore$ Minimum value of $f(x) = 1$.

57. (b) $f(x) = \cos x + \cos\left(\sqrt{2x}\right)$ is 2, attain its maximum at $x = 0$.

There exists no other value of x for which the same value can be attained.

58. (c) $f(x) = \frac{x}{\sin x} \Rightarrow f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = 0$

$\sin x - x\cos x = 0 \Rightarrow x = \frac{\sin x}{\cos x} = \tan x$

In the interval (0, 1), there is no solution for $x = \tan x$

$g(x) = \frac{x}{\tan x} \Rightarrow g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x} = 0 \Rightarrow x = \frac{1}{\sin x\cos x}$

In the interval (0, 1), $g(x)$ is not increasing function.

59. (b) Hence $f(x)$ attains maximum at $x = \frac{1}{4}$.

$f(x) = x^{25}(1-x)^{75}$

$f'(x) = 25x^{24}(1-x)^{75} + 75(1-x)^{74} = 25x^{24}(1-x)^{74}\{(1-x)-3x\} = 25x^{24}(1-x)^{74}(1-4x)$

$f(x)$ is positive for $x < \frac{1}{4}$ and $f(x)$ is negative for $x > \frac{1}{4}$

Hence, $f(x)$ attains maximum at $x = \frac{1}{4}$.

61. (d) Max. value of $(x) = 2 - h$ at left neighbourhood of $x = 0$

$x < 0$ for $-1 \le x < 0$

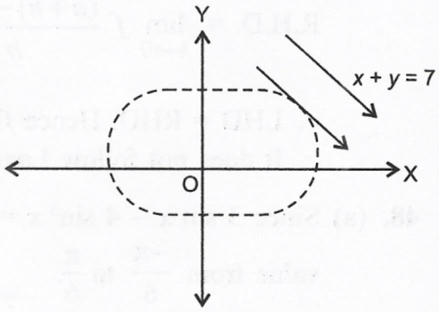
Minimum value of $f(x) = (0 + h)$ at right hand of $x = 0 \Rightarrow x > 0$ for $0 < x \le 1$.

65. (a) $y = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$. Slope (B) = $\frac{dy}{dx} = \frac{4x}{(x^2+1)^2}$

For maxima and minima s. $\frac{ds}{dx} = 0 \Rightarrow \frac{ds}{dx} = \frac{4(x^2+1)^2.1 - x.2(x^2+1)2x}{(x^2+1)^4} = 0$; $x = \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

Slope has maxima at $x = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{-1}{2}$

69. (a) Let us take a point $\left(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta\right)$



On the $\frac{x^2}{6} + \frac{y^2}{3} = 1$.

Slope of normal at P = $\frac{a_2/x_1}{b_2/y_1} = \frac{\sqrt{6}\sec\theta}{\sqrt{3}\mathrm{cosec}\theta} = \sqrt{2}\tan\theta = 1$.

so, $\cos\theta = \sqrt{\frac{2}{3}}$ and $\sin\theta = \frac{1}{\sqrt{3}}$. Hence required point is P (2, 1).

71. (d) $x \in [-1, 2)$, $f'(x) = 6x + 12 > 0$ on $(-1, 2)$

So, f increases on $[-1, 2]$

A point $x = 2$

$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} (37 - x) = 35 \Rightarrow \lim_{x\to2^-} f(x) = \lim_{x\to2^-} (3x^2 + 2x - 1) = 35.$

(f) is continuous at $x = 2$ as well.

As 'f' increases on $[-1, 1]$ and decreases on $[2, 3]$

so, f has a maximum at $x = 2$.

74. (c) $f'(x) = e^x (x - 1)(x - 2)\, dx$

$\therefore f'(x) > 0$ or $e^x > 0$ $(x - 1) > 0$ and $(x - 2) > 0$

$x > 2$ and $x < 1$

75. (c) $x^2 + 3xy - xy - 3y^2 = 0 \Rightarrow x(x + 3y) - y(x + 3y) = 0 \Rightarrow (x + 3y)(x - y) = 0$

$\therefore$ Equation of normal is $(y - 1) = -1(x - 1) \Rightarrow x + y = 2$

It intersects $x + 3y = 0$ at $(3, -1)$

$\therefore$ Curve meet again in the IV quadrant.

76. (a) $f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

$2\beta x^2 + x + \alpha = 0$ has roots -1 and 2.

77. (d) Let $h(f) = f(x) - 2g(x) \Rightarrow h(0) = h(1) = 2$

Hence, using Rolle's theorem

$f'(c) = 0 \Rightarrow f'(c) = 2g'(c)$

78. (a) $f'(c) = \frac{f(3) - f(1)}{3 - 1} \Rightarrow \frac{1}{C} = \frac{\log 3 - \log 1}{2} \Rightarrow C = \frac{2}{\log_c 3} = 2\log_3 e.$

81. (b) $f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot (\cos x - \sin x) = \frac{\sqrt{2}\cos\left(x + \frac{x}{4}\right)}{1 + (\sin x + \cos x)^2}$

$f(x)$ is increasing if $\frac{-\pi}{4} < x + \frac{\pi}{4} < \frac{\pi}{2}$; $\frac{-3\pi}{4} < x < \frac{\pi}{4}$

$\therefore f(x)$ is increasing $\left(\frac{-\pi}{2}, \frac{\pi}{4}\right)$

82. (a) Eliminating θ, we get, $(x - a)^2 + y^2 = a^2$

Hence normal always passes through $(a, 0)$.

84. (i) (b)

(ii) (c) $2b + l = 100$

(iii) (d) Area of garden as a function of x can be represented as

$A(b) = b \cdot l = b(100 - 2b) = 100b - 2b^2$

(iv) (c) $A(b) = 100b - 2b^2$

$A'(b)$ $100 - 4b$

For the area to be maximise $A'(b) = 0$

$100 - 4b = 0 \Rightarrow 4b = 100$; $\Rightarrow b = 25$ ft

(v) (a) Maximum area of the garden

$= 200(25) - 2(25)^2 = 5000 - 1250 = 3750$ sq. ft.